Generalized Graph Signal Processing

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Nanyang Technological University

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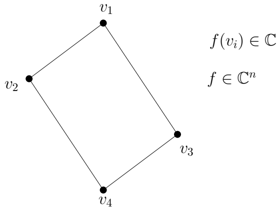
Graph signal processing

- Signal f on a graph $G = (V, E)$: $f : V \mapsto \mathbb{C}$



- Examples: sensor networks, social networks, transportation networks, ...
- Exploits the underlying graph structure (correlations between nodes) to perform signal processing and inference.

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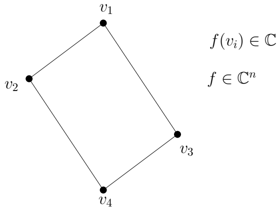

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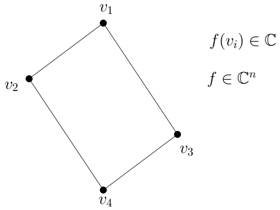

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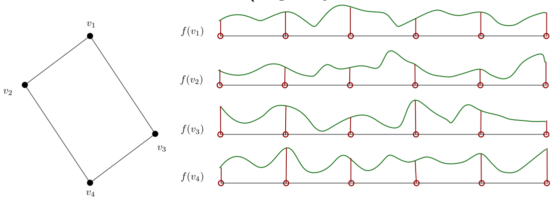
$$\text{GFT: } \hat{f} = \Phi^* f \overset{[1]}{=} (\langle f,\, \phi \rangle_{\mathbb{C}^n})_{\phi \in \Phi}$$

[1] B. Girault, A. Ortega, and S. S. Narayanan, "Irregularity-aware graph Fourier transforms," *IEEE Transactions on Signal Processing*, vol. 66, no. 21, pp. 5746–5761, Nov. 2018.

Time-vertex GSP

- [2]: $f(v,\cdot) \in \mathbb{C}^T$ for each $v \in V$, $T < \infty$, is a discrete time series with common time indices (e.g., synchronous uniform sampling at every vertex).



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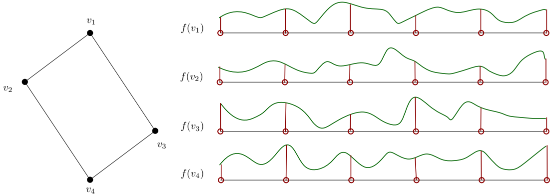

- For each $v \in V$, can apply DFT on $f(v, \cdot)$:

$$\mathsf{DFT}(f(v, \cdot)) = \Xi^* \operatorname{vec}(f(v, \cdot)),$$

where Ξ^* is the DFT matrix.

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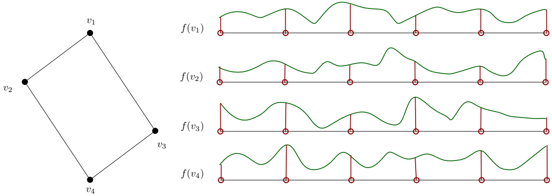

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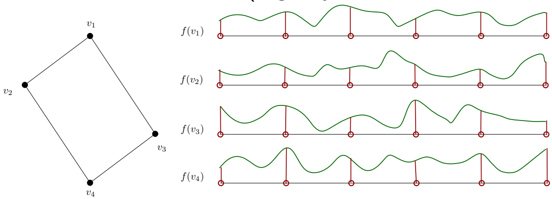

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- A representation in the basis $\Phi \otimes \Xi$.

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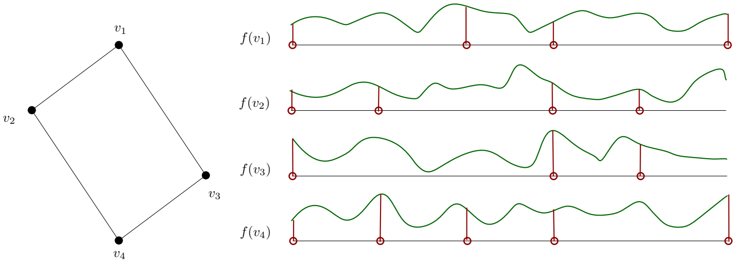

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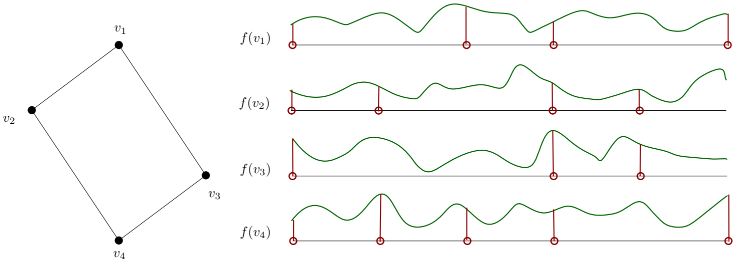

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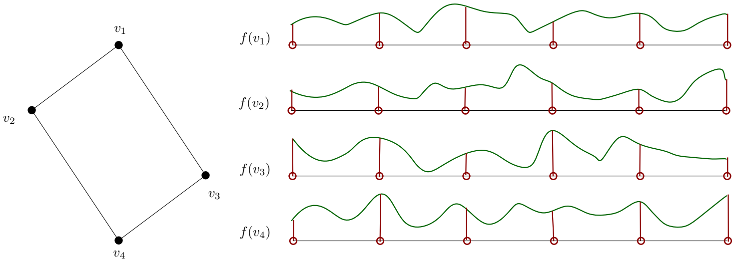

Time-vertex GSP

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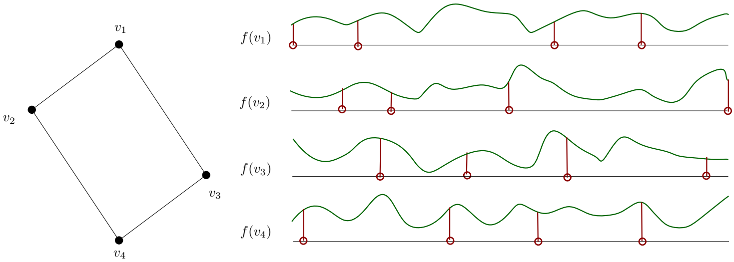
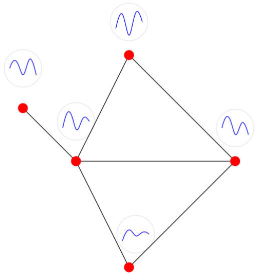
- If vertex domain dimension $= 2$, can reconstruct.
- However, asynchronous sampling (e.g., sensor networks) ... now impossible to reconstruct the signal.



Generalized GSP

- Signal at each vertex is from an infinite dimensional separable Hilbert space $\mathcal{H} = L^2(\Omega, \mu)$.



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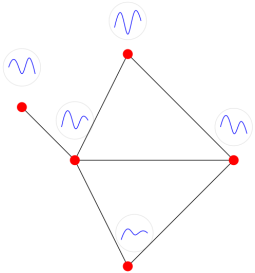

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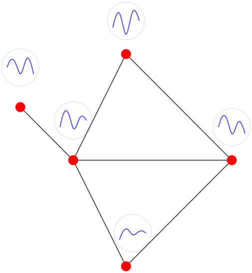

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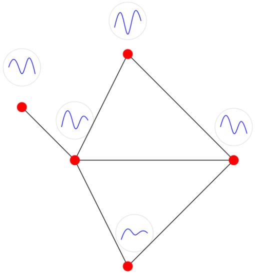
- Non-synchronous sampling (time indices are not same for different vertices).
- Allows joint modeling of $f(v, x)$ over $v \in V$, $x \in \mathcal{H}$.



The rest of this talk...

1. Generalized Graph Signals and $\mathcal{F}$-Transform
2. Sampling Theorem
3. Filtering
4. Conclusion

Details in

F. Ji and W. P. Tay, "A Hilbert space theory of generalized graph signal processing," *IEEE Trans. Signal Process.*, 2019, accepted. [Online]. Available: https://arxiv.org/abs/1904.11655.

Outline

Tensor product space

- Tensor product $\mathbb{C}^n \otimes \mathcal{H} = \left\{ \sum_{i=1}^n v_i \otimes h_i \right\}$ with
  1. (a) $v_1 \otimes h + v_2 \otimes h = (v_1 + v_2) \otimes h$;
  2. (b) $v \otimes h_1 + v \otimes h_2 = v \otimes (h_1 + h_2)$;
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- Inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}^n \otimes \mathcal{H}}$ induced (linearly) by:

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Lemma

$S(G, \mathcal{H})$ is a Hilbert space isomorphic to $\mathbb{C}^n \otimes \mathcal{H}$, $|V| = n$.

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- $f = \sum_{\phi \otimes \xi} \mathcal{F}_f(\phi \otimes \xi) \cdot \phi \otimes \xi$

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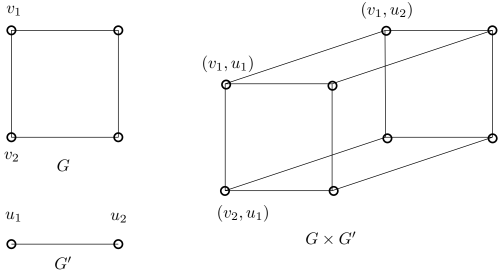

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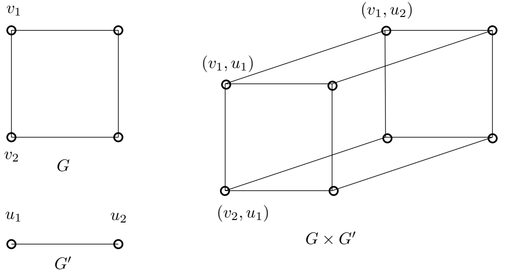

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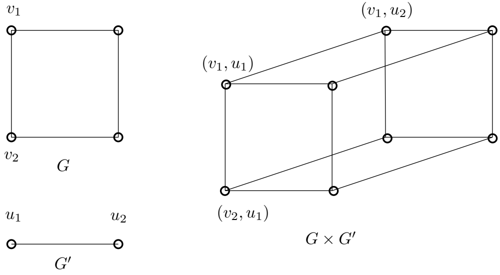
  - G' = path graph: $\mathcal{F}$-transform = TV-transform



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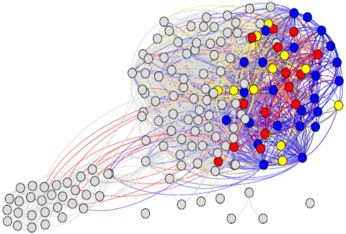
- Fredholm operator

$$Af(x)=\int_{\Omega}K(x,y)f(y)\mathrm{d}\mu(y),$$

Hermitian $K\in L^2(\Omega\times\Omega)\implies A$ compact, self-adjoint.
Choose different kernels for different applications.

Spectral analysis example

- Information propagation over a network: SI, SIR, SIRI



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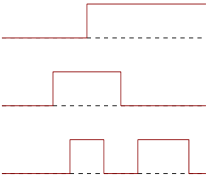

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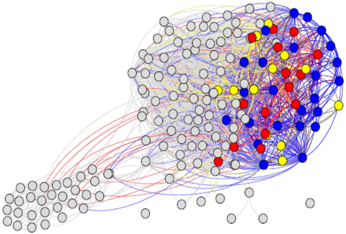

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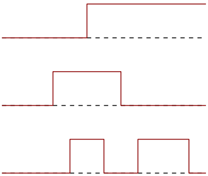

- Signal at each node v: $f(v) \in L^2([0, T])$, step function unbandlimited in time domain.

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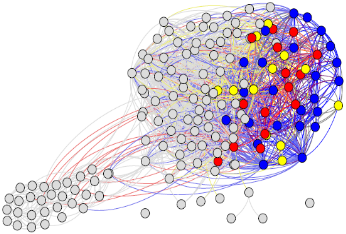

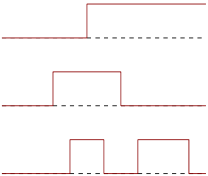

Source: J. McAuley and J. Leskovec. Learning to Discover Social Circles in Ego Networks. NIPS, 2012

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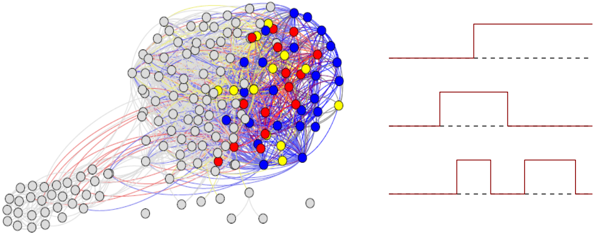

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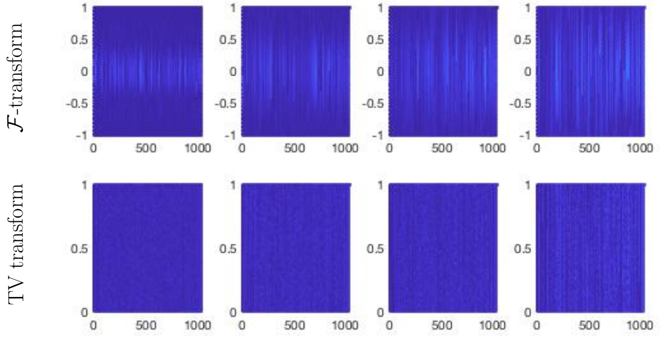

Facebook network, $\lambda_I = 1$: $\lambda_R = 0$, $\lambda_R = 1/5$, $\lambda_R = 1/2$ and $\lambda_R = 1$.

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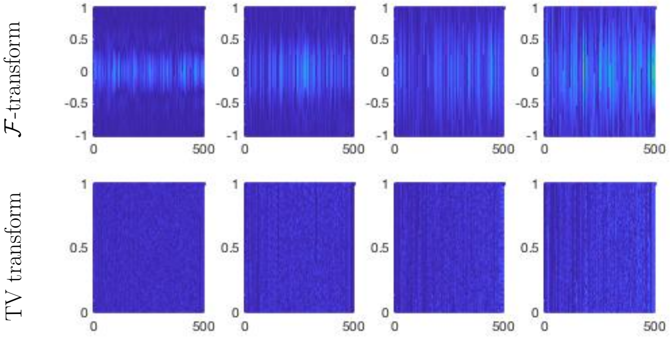

Enron email network, $\lambda_I = 1$: $\lambda_R = 0$, $\lambda_R = 1/5$, $\lambda_R = 1/2$ and $\lambda_R = 1$.

Outline

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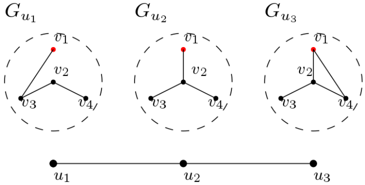

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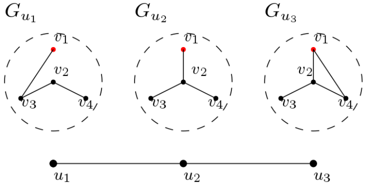

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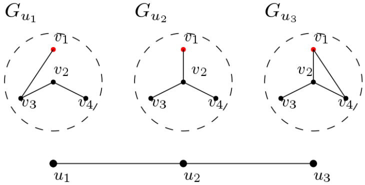

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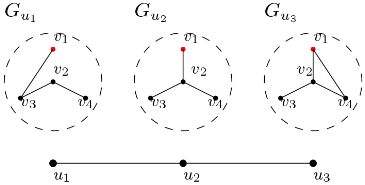

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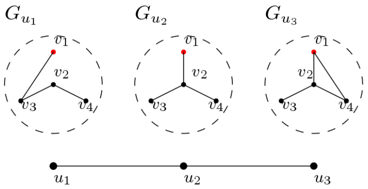

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- "Reconstructible vertex set": can reconstruct whole graph signal at each instant $x \in \Omega$ from signals in this set. $\implies$ linearly independent rows of matrix Φ'.

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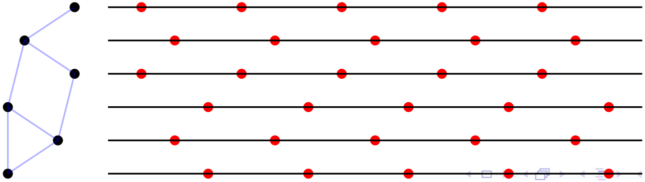

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- But if $\{f(v,\cdot) : v \in V\}$ is bandlimited in graph vertex domain, then only need $\approx 2B/\delta(\Phi')$ for each vertex to recover all signals.

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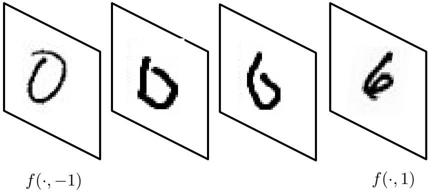

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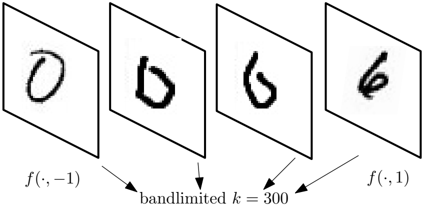

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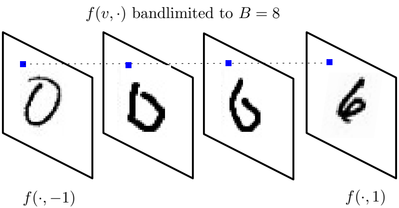

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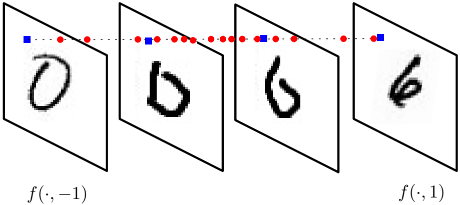

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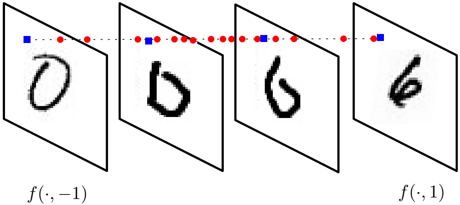

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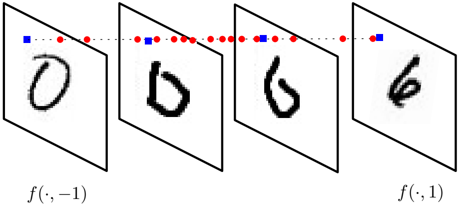

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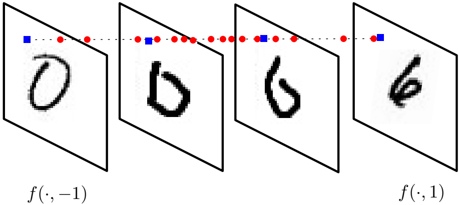

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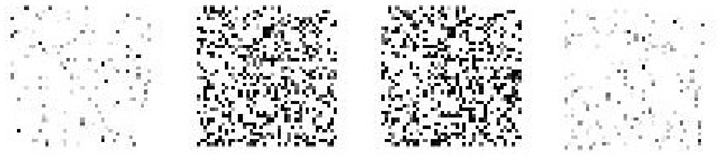

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Shift invariance

- A *filter* is a bounded linear transformation $L : S(G, \mathcal{H}) \to S(G, \mathcal{H})$.
- Tensor product filter $A_G \otimes A(v \otimes h) = A_G(v) \otimes A(h)$ is an example.
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- Not always true, but almost always in practice ...

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- If A_G is the Laplacian matrix, we can restrict to the orthogonal complement of 0-eigenspace of $A_G \otimes A$.

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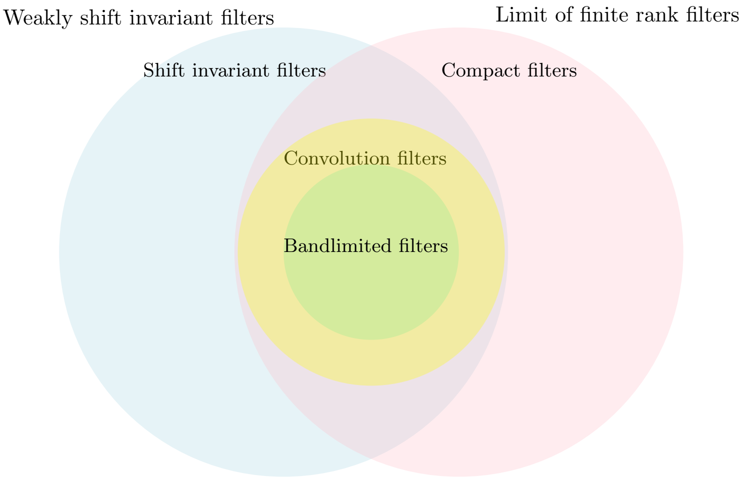
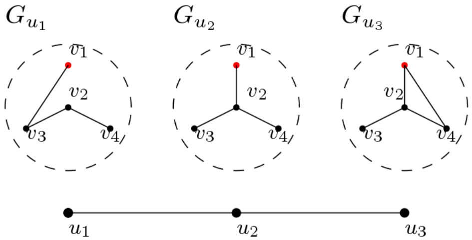
- GSP: all polynomial filters are convolutions.
- Polynomial filter $P(A_G \otimes A)$ with $a_0 \neq 0$ is non-compact, therefore not convolution.

Different classes of filters



Adaptive polynomial filters



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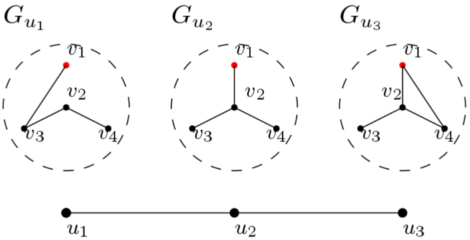

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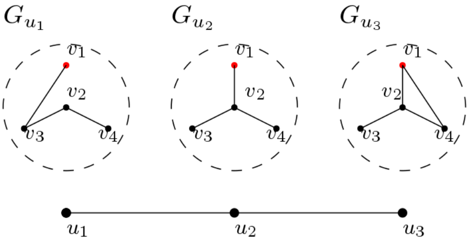

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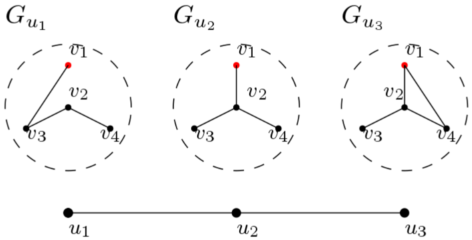

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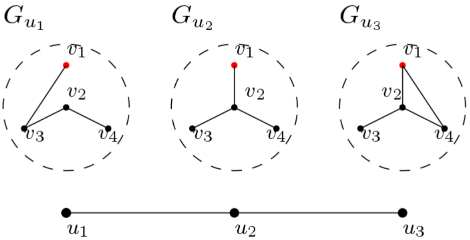

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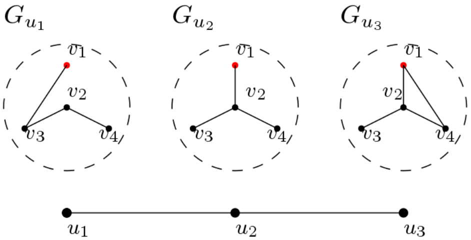

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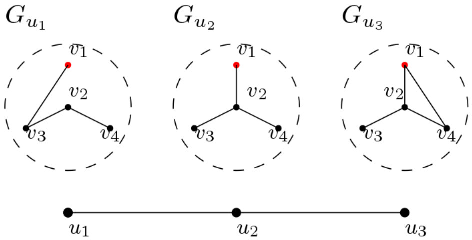

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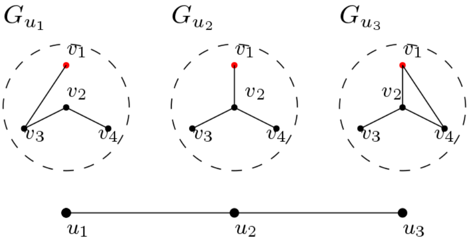

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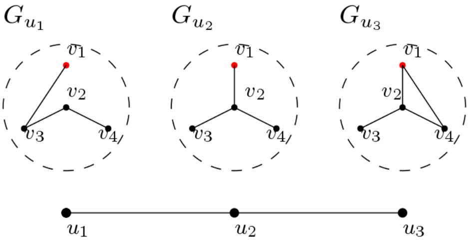

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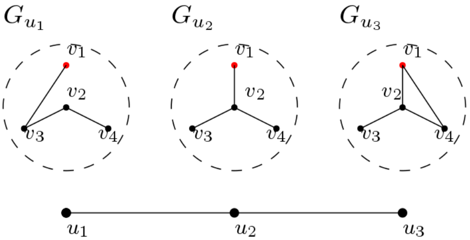

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- Wrong to use GSP on big ambient graph containing all G_{u_i}s.

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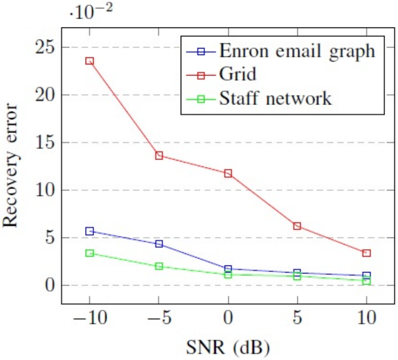

Graphs evolve according to model in [4] (applications in social networks, biological neuron networks, etc.).

[4] J. Ito and K. Kaneko, "Spontaneous structure formation in a network of chaotic units with variable connection strengths," *Phys. Rev. Letts.*, vol. 88, no. 2, p. 028 701, 2002.

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- Shift invariant and weakly shift invariant filters: WSI almost always SI.

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- Other high dimensional extensions: simplicial complexes [7] and hypergraphs [8].

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Thank you!

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